

# The Peculiar Puzzles of Professor Fibbernacho



## TEACHERS' NOTES

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# Teachers' Notes

## Purpose:

- Mathematics is meant to be fun! The Peculiar Puzzles of Professor Fibbernacho brings fun into the classroom while introducing students to a broader range of mathematical topics.

## Who is this book for?

- Middle School (Years 5 – 8) students with a keen interest in mathematics, codes and secrets, and who enjoy a challenge. As students work through the book, the puzzles increase in mathematical complexity. The early puzzles are suitable for the whole class, and subsequent puzzles are more appropriate for highly able children.

## Special Features:

- Each puzzle is complete and independent. Early finishers can move on to the next puzzle while the rest of the group may work on one puzzle per lesson. Each puzzle will take an average student about 45 minutes to complete.
- The puzzles cover many aspects of mathematics as well as a wide range of topics, such as history and general science. Each puzzle introduces a new topic and has a new puzzle to solve. Each puzzle has certain clues, and a coded message. Students use the mathematics they have just learnt to help crack the code, and to solve the puzzle.
- Students use a new code for each puzzle. Some of these include substitution ciphers, Morse code and binary code.
- Black Line Masters are available on the G.A.T.E.WAYS website. Students can share a book with a group, while individually answering the exercises and decoding the secret messages.

## Curriculum Area: The mathematics covered include:

- Sets and Numbers
- Prime Factorization
- Sequences
- Prime Numbers
- Fibonacci Numbers
- Golden Ratio
- Binary Numbers
- Square Roots
- Square Numbers
- Triangular Numbers
- Perfect Numbers
- Infinity
- Clock Arithmetic
- Topology

## What does this book contain?

- A brief introduction by Professor Fibbernacho and his steadfast companion, Sophie
- Ten adventures each including
  - A perplexing puzzle
  - An introduction to the mathematics pertinent to the puzzle
  - Examples and exercises to complete
  - A code to decipher, which contains the answer to the puzzle
- **Advanced questions** and **further research** for students who may want further challenges
- **Did You Know** section, containing further information about the history and importance of mathematics
- **Solutions** to the set exercises and coded messages
- **Black Line Masters** and **Teachers' notes** available on the G.A.T.E.WAYS website

## Why is this book so important?

- Each puzzle introduces the student to a concept of mathematics that they may not have come across, or have not explored in much detail.
- The material contains fundamental topics and concepts that play a key role in higher level mathematics. They are inspiring, creative, challenging and fun.
- In general education there is unfortunately a vast body of mathematics that is often not explored deeply enough to give students a feel for the magic and mystery of mathematics.

## Further Research:

- The **Further Research** section in each puzzle is designed to be a stepping stone for highly able students who want to explore the related topic in more depth. Most of the questions given are open ended questions, designed to give a starting direction for the student to begin their own research.
- Although answers are not given for this section, a guide has been included here to assist teachers in their explanations.

## Puzzle One: Game, Set, Math

A **set** is a collection of similar objects or elements. An example of a set of numbers is all the even numbers from two to ten. This is represented as follows:  $\{2,4,6,8,10\}$ , where the curly brackets  $\{\}$  denote a set. The order of the numbers (or elements) in the set is not important. The same set could be written:  $\{10,2,6,4,8\}$ . A set can have a finite number of objects or an infinite number of objects. For example, the set of whole numbers from one to five is  $\{1,2,3,4,5\}$  and is a finite set. The set of prime numbers is  $\{2,3,5,7,11,\dots\}$  and is an infinite set. The three dots represent the numbers continuing to infinity, following the same pattern. Most sets are named using a letter, for example  $\mathbf{E} = \{2,4,6,8,10,\dots\}$  and  $\mathbf{O} = \{1,3,5,7,9,\dots\}$ .

The main sets of numbers that we use in mathematics are as follows:

The **Natural numbers** are all the whole numbers, beginning at one.  $\mathbf{N} = \{1,2,3,4,5,\dots\}$

The **Integers** are all the positive and negative whole numbers, and zero.  $\mathbf{Z} = \{\dots-3,-2,-1,0,1,2,3,\dots\}$

The **Rational numbers** are all the integers and all the fractions.  $\mathbf{Q} = \{m/n \text{ where } m \text{ and } n \text{ are integers and } n \text{ cannot equal zero}\}$

The **Real numbers** are all the rational numbers and all the numbers that cannot be written as a fraction, for example  $\pi$ ,  $\sqrt{2}$ ,  $\sqrt{3}$  and so on. This set is denoted by  $\mathbf{R}$ .

The **Complex numbers** are another set of numbers. This set is denoted by  $\mathbf{C}$ . These numbers are beyond the scope of this book.

It is worth noting that all  $\mathbf{N}$  are in  $\mathbf{Z}$ , all  $\mathbf{Z}$  are in  $\mathbf{Q}$  and all  $\mathbf{Q}$  are in  $\mathbf{R}$ .

A **permutation** is a rearrangement. The **code** in this puzzle is a **permutation cipher**; rearranging the letters in each word will decode the message.

Binary code uses the digits zero and one. Text and computer instructions are converted into binary code, as it is a relatively simple way of storing information. Binary codes are not a new concept; they have been around since about 100 BC. There are many other examples of number codes. Some of these we use frequently are library codes, bar codes, ISBN and so forth.

## Puzzle Two: Top That

A **polygon** is a two dimensional shape with straight sides. Examples of these include squares, rectangles, hexagons and kites. A circle is not a polygon. A **regular polygon** has sides of equal length and equal angles. A **polyhedron** is a three dimensional shape whose faces are made of polygons. A **regular polyhedron** is made up of regular polygons. A **tetrahedron** is made of four triangular faces. A **square based pyramid** is made of four triangular faces joined by a square base (five faces altogether). An **octahedron** is simply two square based pyramids joined together at their bases (eight faces altogether). A **buckminsterfullerene** is made of 20 hexagons and 12 pentagons, making 32 faces altogether. An example of this is a soccer ball.

If you take a strip of paper and join two ends, you get a cylinder. If you take a cylinder and join the top and bottom together, you end up with a donut shape, called a **torus**. If you take a strip of paper and put a twist in it before joining the ends, you get a **Mobius strip**. If you take a Mobius strip and join the top and bottom together you get a **Klein bottle**.

If you count the number of  $\mathbf{C}$  = corners,  $\mathbf{F}$  = faces and  $\mathbf{E}$  = edges of a polyhedron, and calculate  $\mathbf{C} + \mathbf{F} - \mathbf{E}$  you will always get the answer two. This formula does not work for polygons.

**Euclid of Alexandria** (300BC) is known as the 'Father of Geometry'. His extensive collection of works, referred to as 'The Elements' consists of thirteen books covering definitions, axioms, theorems and proofs of many aspects of geometry and number theory.

**Morse code** in this puzzle is an example of a **substitution cipher**; the letters are substituted for symbols.

A sculpture of a Mobius strip is found in the Baltimore Museum of Art, in America. There are many other galleries around the world that have such sculptures. The artist M C Escher (1898 – 1972) was a Dutch graphic artist. Some of his famous works include 'drawing hands', 'ascending and descending' and 'symmetry'. The surface of the Earth is a two dimensional surface (any point on the surface of the Earth can be described by two numbers – corresponding to longitude and latitude). This surface exists in three dimensions, as it is a curved surface. A Mobius strip is also a two dimensional surface that exists in three dimensions. A Klein bottle is a two dimensional surface that exists in four dimensions.

## Puzzle Three: Fe Fi Fo Fum

A **sequence** is a pattern of numbers written in a list. An example of this is the sequence of numbers starting from one and adding one each time. (1, 2, 3, 4, 5, ...). A sequence is different from a set, as the order of numbers in a sequence is important. Round brackets () are often used to denote a sequence to distinguish it from a set. Sequences can contain a finite or an infinite number of numbers.

The **Fibonacci sequence** begins with the numbers zero and one. To get to the next number you add the two previous numbers together. Thus the next number in the sequence is  $0+1=1$ . The next number is  $1+1=2$ . Thus the sequence is (0, 1, 1, 2, 3, 5, 8, 13, 21, ...). The Fibonacci numbers come up quite a lot in nature. If you count the spirals in a pinecone you will find in one direction there are eight and in the other direction there are thirteen. Both of these numbers are Fibonacci numbers. The ratio of two consecutive Fibonacci numbers is close to an irrational number known as **phi**. Phi is roughly equal to 1.6180339887498... You also get close to phi if you divide the side lengths of an A4 piece of paper, or the side lengths of a credit card, or the side lengths of the bricks used in ancient Roman buildings.

The sequence 1, 3, 6, 10, 15, 21, 28, ... is made by starting at one and adding two to get three, then adding three to get six, then adding four to get ten, and so on. These numbers are known as the **Triangular numbers** because they can each be represented by dots in a triangle.

The sequence 2, 3, 5, 8, 12, 17, ... is similar to the one above but beginning at two.

The sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, ... is the Fibonacci sequence described above, but beginning at one.

The sequence 1, 4, 9, 16, 25, 36, 49, 64, ... is made by squaring the numbers 1,2,3,4,5,6,7,8,... These are known as the **Square numbers**.

The **code** in this puzzle is also a **substitution cipher**, but this time the letters are substituted for other letters. To find the **key** for the code, we use the Fibonacci sequence from the exercises. This gives us the sixth Fibonacci number (eight) minus the fourth Fibonacci number (three) which is  $8-3=5$ . Thus the key is the number **five**.

The alphabet the students are required to write looks like this:

**A B C D E F G H I J K L M N O P Q R S T U V W X Y Z**  
**F G H I J K L M N O P Q R S T U V W X Y Z A B C D E**

To decode, the students need to find which way the code works, either up or down. If the coded message was

**M J Q Q T**

and you looked at the alphabet on the top row first, you would find **M** corresponds to **R**, **J** to **O**, **Q** to **V** and **T** corresponds to **Y**. This means your decoded message is ROVVY, which is nonsense. If you look at the alphabet on the bottom row first, you find M corresponds to H, and so on to make the word **HELLO**.

The coded message asks students to divide a number in the Fibonacci sequence by the number sitting to the left. This would give them thirteen divided by eight, or eight divided by five, and so on. The larger the numbers they choose from the sequence, the closer they get to 1.6180339... which is phi.

The Fibonacci sequence begins with the numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... . There are many examples in nature where these numbers are found. The number of petals on many flowers will be a number in the Fibonacci sequence. The number of spirals in each direction on a sunflower will vary according to the size and variety of the flower, but the numbers found will always be two consecutive numbers in the Fibonacci sequence, as will the number of spirals on a pinecone, or a pineapple.

## **Puzzle Four: Primate Helpers**

The **Ishango** bone is the femur bone from a monkey, discovered in Africa in 1960. It dates back to 6500 BC. It has three columns of four groups of notches inscribed on it. In one column are the numbers 11,13,17,19 which are four consecutive prime numbers.

**Prime numbers** are used in coding messages in all aspects of online communication. The mathematical process for this is quite involved and well beyond the scope of this book, but uses the fact that some very large numbers are incredibly difficult to factorize into their prime factors.

To find if a number  $N$  is prime we only need to check if the prime numbers from 2 to  $\sqrt{N}$  divide this number. For example 101 is not divisible by 2, 3, 5 or 7. We can now say with certainty that 101 is a prime number, as  $\sqrt{101}$  is slightly larger than 10, thus we do not need to check if 11 divides 101. The reason for this is if a number larger than  $\sqrt{N}$  divided it, the other divisor would be smaller than  $\sqrt{N}$  and we would have already found that divisor.

The number one is not a prime number.

The **key** for the **code** in this puzzle is the number **ten**. This means that the alphabet the students are required to write looks like this:

**A B C D E F G H I J K L M N O P Q R S T U V W X Y Z**  
**K L M N O P Q R S T U V W X Y Z A B C D E F G H I J**

Eratosthenes (276-194 BC) was the first person to accurately measure the circumference of the Earth, using shadows and the summer solstice. He also calculated the distance from the Earth to the Sun, the distance of the Earth to the Moon, and calculated the tilt of the Earth's axis. He created a catalogue of stars, which included over 600 stars, and came up with the idea of a calendar using leap years, similar to the method we use today. There is a crater on the Moon named after Eratosthenes.

## Puzzle Five: Shaking Hands

The maximum number of handshakes that can occur between differing numbers of people creates a pattern. This pattern can easily be seen by counting: Two people have **one** handshake, three people have **three** handshakes, four people have **six** handshakes, five people have **ten** handshakes, and so on. We can see that the number of handshakes is the sequence of **triangular numbers** we saw in Puzzle Three.

The prime factorization of a number simply requires finding the prime numbers that multiply together to produce this number. The prime factorization of 33 is  $11 \times 3$ . The prime factorization of 16 is  $2 \times 2 \times 2 \times 2$ .

The code in this puzzle is another substitution cipher, but this time the letters are substituted for numbers. The numbers begin at 00 and end at 25.

Euclid of Alexandria (Egypt) lived around 300 BC. It is unknown exactly when and where he was born, but he lived some of his life in Athens, and it was most likely here that he wrote his epic works, *The Elements*.

## Puzzle Six: Tower Of Hanoi

The **Tower of Hanoi** is a puzzle consisting of three rods, and a number of hoops of different sizes. The hoops are set up on one rod, from largest at the bottom to smallest at the top. The idea is to move all the hoops across to a different rod, using the smallest number of moves possible. But you can only move one hoop at a time, and you cannot place a larger hoop on a smaller one. Following these rules we find that starting with two hoops, it will take a minimum of three moves to complete the puzzle. The game itself is not what is important in this puzzle, but finding the pattern of the number of moves is. By experimentation we can find that one hoop takes one move, two hoops take three moves (twice the moves for one hoop, plus one more move), three hoops take seven moves (twice the moves for two hoops, plus one more move), four hoops takes fifteen moves, and so on. The sequence of numbers created by the number of moves is 1, 3, 7, 15, 31, ....

**Perfect numbers** are numbers whose factors (except the number itself) add to that number. **Six** is a perfect number as  $1+2+3=6$ . 8 is not a perfect number as  $1+2+4=7$ . **28** is a perfect number as  $1+2+4+7+14=28$ . The next perfect number after 28 is 496. Perfect numbers are quite rare, and all the known ones are even.

**Friendly numbers** are pairs of numbers whose factors (except the number itself) add to each other. 220 and 284 are friendly numbers. The factors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110. Adding these factors gives us  $1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$ . The factors of 284 are 1, 2, 4, 71 and 142. Adding these factors gives us  $1 + 2 + 4 + 71 + 142 = 220$ . The numbers 1184 and 1210 are also friendly numbers. There are many patterns and relationships that occur with these numbers. Every perfect number that we know of is also a triangular number. Two consecutive triangular numbers add together to make a square number.

**Pentagonal numbers** are numbers that are made from pentagons, rather than from squares or triangles. The sequence of triangular numbers is 1, 3, 6, 10, 15, 21, ... (adding one more each time to the difference). The sequence of square numbers is 1, 4, 9, 16, 25, ... (adding two more each time to the difference). The sequence of pentagonal numbers is 1, 5, 12, 22, 35, ... (adding three more each time to the difference). We can continue this thinking on to create hexagonal numbers, heptagonal numbers, and so on.

**Pythagoras** believed that only whole numbers and fractions existed. It is said that one of his students mysteriously disappeared after showing Pythagoras his proof of the existence of **irrational numbers**. (Irrational numbers form part of the Real numbers. They are all the Real numbers that cannot be written as fractions.)

All perfect numbers discovered are even, but it is not known if odd perfect numbers exist. Perfect numbers were once thought to have numerological properties. Today they share a connection with prime numbers.

## Puzzle Seven: Hotel California

One of the most profound concepts in mathematics is that of **infinity**. Many students try to do arithmetic (add, subtract, multiply, divide) on infinity. This fails as infinity is not a number, it does not fit into any of the sets of numbers we described in Puzzle One. This is best demonstrated by taking a list of the Natural numbers (which go on to infinity) and separating the even numbers from the odd numbers. Now we have two lists, each with an infinite number of numbers in them. This would give us infinity plus infinity equals infinity, which makes no sense at all. We simply cannot do arithmetic with infinity.

The mathematician David Hilbert (1862-1943) came up with the scenario of a hotel with an infinite number of rooms. If all the rooms are booked, the hotel manager can still take one more booking by moving every guest up one room, leaving room number one free. The hotel manager can also take an infinite number of new guests by moving every booked guest into the room twice their original room number. Thus the guest in room one goes to room two, the guest in room two goes to room four, the guest in room three goes to room six, and so on. This leaves all the odd numbered rooms (of which there are an infinite number) free for our infinite number of new guests.

The **code** in this puzzle combines a shift cipher with the substitution cipher that we have been using. The key the students need is the number **two**, and the code they write is as follows: **A = 02, B = 03, ..., Y = 00, Z = 01**.

Dividing a number by zero is the same as trying to find a number that you could multiply by zero to get a non zero answer. As an exercise, students can divide a given number by numbers that get smaller and smaller (ie closer and closer to zero) to see what happens to their answer. If you do this, you find the answers you get become larger and larger, closer and closer to infinity.

## Puzzle Eight: This Sentence Is False

A **paradox** is a statement or story that contradicts itself. An example of this is the statement, "I always lie". The mathematician Bertrand Russell (1872-1970) came up with the idea of the **barber's paradox**. There is a town where all the men are shaved. The men fit into two distinct groups – those that shave themselves, and those who are shaved by the barber. The question asks what group the barber fits into. The paradox lies in the assumption that such a town could ever exist.

**Binary numbers** are a system of numbers using only the digits zero and one. Our decimal system uses ten digits, from zero to nine. Comparing the two systems, we see that our decimal system uses columns of (at least) 1000s, 100s, 10s and units. We also know  $1000 = 10 \times 10 \times 10 = 10^3$ ,  $100 = 10 \times 10 = 10^2$ ,  $10 = 10^1$  and  $1 = 10^0$ . If we consider the number 4609, we can break this down using our decimal columns:

$$4609 = 4 \times 1000 + 6 \times 100 + 0 \times 10 + 9 \times 1$$

Using this same method in the binary system, we have columns of (at least) 8s, 4s, 2s and units. We can also write  $8 = 2 \times 2 \times 2 = 2^3$ ,  $4 = 2 \times 2 = 2^2$ ,  $2 = 2^1$  and  $1 = 2^0$ . If we consider the number thirteen, we can break this down using our binary system in exactly the same way as we used our decimal system.

$$13 = 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1$$

This means our binary representation of thirteen is **1101**.

This same thinking can be applied to the ternary system, using the digits zero, one and two. In this system we use the columns (at least) 27s, 9s, 3s and units. We can also write  $27 = 3 \times 3 \times 3 = 3^3$ ,  $9 = 3 \times 3 = 3^2$ ,  $3 = 3^1$

and  $1 = 3^0$ . If we consider the decimal number sixteen, we can break this down using our ternary system in exactly the same way as above.

$$16 = 1 \times 9 + 2 \times 3 + 1 \times 1$$

Our ternary representation of sixteen is **121**.

The **code** in this puzzle is a substitution cipher, but now the alphabet is coded by a sequence of five digits which represent the binary numbers from 0 to 25.

We can represent the binary numbers on our fingers by using our fingers as the columns, where our little finger = units, ring finger = 2s, middle finger = 4s, index finger = 8s and thumb = 16s. The first few binary numbers would be as follows:

Closed fist = 0

Little finger up = 1

Little finger down, ring finger up = 2

Little finger up, ring finger up = 3

Little finger down, ring finger down, middle finger up = 4

Little finger up, ring finger down, middle finger up = 5

Little finger down, ring finger up, middle finger up = 6

Little finger up, ring finger up, middle finger up = 7

Little finger down, ring finger down, middle finger down, index finger up = 8

and so on.

## Puzzle Nine: Three Triangles

Carl Gauss (1777 – 1855) is known as the Prince of Mathematics. He is one of the most well known mathematicians. He stated that every whole number is the sum of one, two or three triangle numbers.

The **hailstone** problem was first proposed in 1937. It is an unsolved problem in mathematics as it is not known whether or not all numbers put in to the hailstone sequence lead to the sequence 4, 2, 1.

The **code** in this puzzle is the same binary substitution as for Puzzle Eight, but this time the students need to put in their own word breaks. Once the message is decoded, students get to work through Gauss's idea by finding triangular numbers that add to each number, for example  $5 = 3 + 1 + 1$ ,  $6 = 3 + 3$ , and  $7 = 6 + 1$  (or  $3 + 3 + 1$ ).

The hailstone problem is also known as the Collatz conjecture, which states that the sequence of numbers will eventually reach the number one, regardless of the number that you begin with. This has not been proved, nor has any counter example been discovered to show it is false. The Collatz conjecture was first proposed in 1937. It is an example of a problem in mathematics that requires some deep mathematical thinking in order to find a solution. Students who enjoy working with number patterns will tackle this question with interest.

## Puzzle Ten: The Bridges Of Konigsberg

**The Seven Bridges of Konigsberg** is a mathematical problem based on the seven bridges that spanned the Pregel River in Konigsberg (now Kaliningrad, Russia). Residents used to wonder if there was a way to cross the seven bridges once and only once, and thus cross from the mainland to the two islands that were joined by the bridges. In 1736 Leonhard Euler proved that there was no path that they could take that would do this.

**Clock arithmetic** is an important part of mathematics. It deals with the remainder left after division. It is easiest to see how this works by using a regular clock. If it is 4 o'clock and we want to add 10 hours, we can easily see that it will be 2 o'clock. This gives us the sum  $4 + 10 = 2$ , in our 12 o'clock arithmetic. This is the same as calculating  $4 + 10 = 14$  and then finding the remainder of 14 divided by 12, which is 2. There is nothing magical about the number 12, we can construct clocks of any number, and perform arithmetic in this o'clock.

Using a 9 o'clock, we find:

$6 + 5 = 11$ , and 11 divide 9 gives **2** remainder. Hence  $6 + 5 = 2$ .

$8 + 13 = 21$ , and 21 divide 9 gives **3** remainder. Hence  $8 + 13 = 3$ .

Using a 12 o'clock, we find:

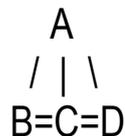
$5 + 9 = 14$ , and 14 divide 12 gives **2** remainder. Hence  $5 + 9 = 2$ .

$6 + 7 = 13$ , and 13 divide 12 gives **1** remainder. Hence  $6 + 7 = 1$ .

$9 + 14 = 23$ , and 23 divide 12 gives **11** remainder. Hence  $9 + 14 = 11$ .

$14 + 11 + 8 = 33$ , and 33 divide 12 gives **9** remainder. Hence  $14 + 11 + 8 = 9$ .

After decoding the secret message, students work through Euler's proof of the above problem. They construct a diagram as follows, where A, B, C and D represent the separate land masses, and the lines between them represent the bridges connecting each land mass.



The above diagram shows three bridges exiting land mass A, three bridges exiting land mass B, three exiting land mass D, and five exiting land mass C. Euler found that to be able to cross the seven bridges once only, there must be an odd number of bridges exiting only two of the land masses, and the rest of the land masses must have an even number of bridges exiting them. This is clearly not the case for the Seven Bridges of Konigsberg.

This concept is easier to see with a simpler example. Imagine we have three land masses, A, B and C, and one bridge connecting A and B, one connecting A and C, and two connecting B and C. There are two bridges exiting land mass A, three exiting land mass B and three exiting land mass C. This fits with Euler's description, and we can cross all these bridges once only. We can take different paths, such as C-B-A-C-B or B-A-C-B-C and we cross each bridge once and only once. Notice that we have to choose our starting point carefully, as this works only if we choose a land mass with an odd number of bridges exiting it (land mass B or C). It does not work if we choose land mass A as our starting point.



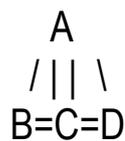
Another example shows that there can be an even number of bridges coming off all land masses, and this gives us a path which will bring us back to our original land mass.



Thus there are two situations which work.

1. If there is an odd number of bridges exiting only two land masses, and the rest have an even number of bridges exiting them.
2. If there is an even number of bridges exiting all the land masses. This situation gives us a closed path, meaning we end up back at the land mass that we started from.

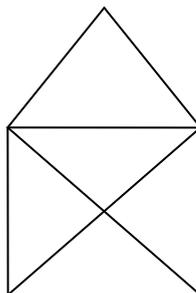
We can alter the diagram of the Seven Bridges of Konigsberg to a situation where there are four bridges exiting land mass A, three bridges exiting land masses B and D, and six bridges exiting land mass C. This now gives us a situation that works.



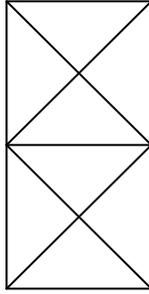
One of the paths we could take is B-A-C-B-C-D-A-C-D. Notice that we began and ended on land masses with an odd number of bridges exiting them.

If we wanted to get back to the land mass that we started from, we would need to alter our diagram again. We could take away one of the bridges connecting B and C, and one of the bridges connecting C and D, and this would give all the land masses an even number of bridges exiting them. We could then travel over a selection of paths, ending at the land mass that we began our journey on. One example is A-D-C-A-B-C-A. Our path would be a closed path.

A similar example can be found when trying to draw a line figure without redrawing over any lines, and without lifting one's pen from the paper. If we draw a figure in which this is possible, such as the figure below, we can see that the two bottom vertices (equivalent to Euler's land masses) have an odd number of lines (bridges) travelling from them, and the two top vertices have an even number of lines travelling from them.



In the second diagram, all the vertices have an odd number of lines coming away from them; hence it is impossible to create this diagram without lifting one's pen from the paper.



Clock arithmetic is an important tool in higher level mathematics. It is closely related to the remainder after division. Many problems in mathematics have been solved with the help of clock arithmetic (or modular arithmetic, as it is called) including Fermat's Last Theorem. Clock arithmetic is also a concept that we use frequently in our everyday lives. Time, weeks, months, any numbered event that follows a cycle, involves clock arithmetic.